Functor calculus and chromatic homotopy theory

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Ordinary calculus

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth (that is, infinitely differentiable) function. Then, for each point $x_0 \in \mathbb{R}$, there exists a real number $s = f'(x_0)$ such that f is closely approximated by the linear function $x \mapsto f(x_0) + s(x - x_0)$ in a small neighborhood of x_0. Linear functions are usually much more tractable than non-linear case.

Furthermore, one can approximate function f by polynomials with degree no more than n. The best choice is n-th Taylor approximation to f (at the point $0 \in \mathbb{R}$), denoted as

$$P_n(f)(x) := c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

$$c_k = \frac{f^{(k)}(0)}{k!}$$

The main ideal is to generalize the linear or polynomial approximations to a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$.

Analogies in calculus of functors

Ordinary Calculus	Functors Calculus
Smooth manifold M	Compactly generated ∞-category $\mathcal{C}$
Point $x \in M$	Object $X \in \mathcal{C}$
Real vector space $\mathbb{R}^n$	Stable ∞-category
Real numbers $\mathbb{R}$	∞-category Sp of spectra
Smooth function $f : M \to N$	Functor $F : \mathcal{C} \to \mathcal{D}$ which preserves filtered colimits
Linear map of vector spaces	Exact functor between stable ∞-categories
Tangent space $T_x M$ at x	∞-category of spectrum objects $Sp(\mathcal{C}_{/X})$
Polynomial approximation of f	Excisive functors of F

n-cube

We will require the ∞-category below admit limits and colimits.

Definition

An n-cube in an ∞-category $\mathcal{C}$ is a functor $X : N(P(I)) \to \mathcal{C}$, where $P(I)$ is the poset of subsets of some finite set I of cardinality n. An n-cube X is **cartesian** if the canonical map

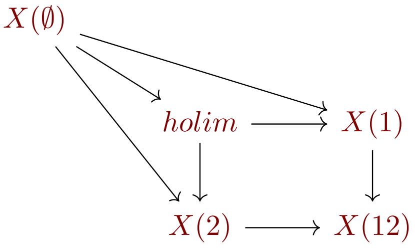
$$X(\emptyset) \to holim_{\emptyset \neq S \subset I} X(S)$$

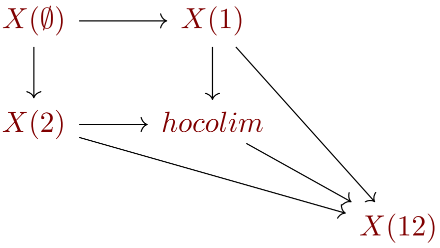
is an equivalence, and **cocartesian** if

$$hocolim_{S \subsetneq I} X(S) \to X(I)$$

is an equivalence. When $n = 2$, these notions are exactly **pullback** and **pushout**. We say that an n-cube X is **strongly cocartesian** if every 2-dimensional face is a pushout. A strongly cocartesian n-cube is also cocartesian if $n \geq 2$.

n-cube





n-excisive functor

Definition (n-excisive)

Let $\mathcal{C}$ be an ∞-category that admits pushouts. A functor $F : \mathcal{C} \to \mathcal{D}$ is n-excisive if it takes every strongly cocartesian $(n+1)$-cube in $\mathcal{C}$ to a cartesian $(n+1)$-cube in $\mathcal{D}$. We will say that F is polynomial if it is n-excisive for some integer n.

Let $Fun(\mathcal{C}, \mathcal{D})$ be the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$, and let $Exc_n(\mathcal{C}, \mathcal{D})$ denote the full subcategory whose objects are the n-excisive functors.

Example

A functor $F : \mathcal{C} \to \mathcal{D}$ is 1-excisive if and only if it takes pushout squares in C to pullback squares in D. The prototypical example is

$$X \to \Omega^\infty(E \wedge X).$$

Polynomial approximation

In ordinary differential calculus: given a smooth function $f : \mathbb{R} \to \mathbb{R}$ and a real number x, there is a unique "best" degree $\leq n$ polynomial that approximates f in a "neighbourhood" of x. To transfer this idea to the calculus of functors, we need to be able to compare the values of functors on objects in $\mathcal{C}$. So we consider slice ∞-category $\mathcal{C}_{/X}$.

Definition

We say that functors $F : \mathcal{C} \to \mathcal{D}$ admit n-excisive approximations at X in C if the composite

$$Exc_n(\mathcal{C}_{/X}, \mathcal{D}) \hookrightarrow Fun(\mathcal{C}_{/X}, \mathcal{D}) \to Fun(\mathcal{C}, \mathcal{D})$$

has a **left adjoint**. When it exists, the n-excisive approximation to $F : \mathcal{C} \to \mathcal{D}$ is another functor from $\mathcal{C}$ to $\mathcal{D}$, which we simply denote by $P_n F$.

Existence of n-excisive approximations

Theorem (Goodwillie (for Top and Sp), Lurie)

Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and suppose that $\mathcal{C}$ has pushouts, and that $\mathcal{D}$ has sequential colimits, and finite limits, which commute. Then functors $\mathcal{C} \to \mathcal{D}$ admit n-excisive approximations at any object $X \in \mathcal{C}$.

Example (0-excisive)

The 0-excisive approximation to F at X is equivalent to the constant functor with value $F(X)$. $P_0F(X) = F(X)$.

Example (1-excisive of identity on Top)

The 1-excisive approximation to the identity functor I on based spaces Top_* is stable homotopy functor

$$P_1I(X) \simeq \Omega^\infty \Sigma^\infty X = Q(X).$$

We will see that this means the first derivative of I is sphere spectrum $\partial_1 I \simeq \mathbb{S}^0$.

Inclusion of excisive functors

Proposition

Let S be a finite set and T a finite subset of S. Suppose we are given an S-cube $X : N(P(S)) \to \mathcal{C}$ in an ∞-category $\mathcal{C}$. Then:

1. *If X is strongly coCartesian, then every T-face of X is strongly coCartesian.*
2. *If every T-face of X is Cartesian, then X is Cartesian.*

Corollary

Let $F : \mathcal{C} \to \mathcal{D}$be a functor between ∞-categories. Assume that $\mathcal{C}$ admits finite colimits and $\mathcal{D}$ admits finite limits. If F is n-excisive, the F is m-excisive for each $m \geq n$. Hence, we have inclusions of subcategories:

$$Exc_0(\mathcal{C}, \mathcal{D}) \subset Exc_1(\mathcal{C}, \mathcal{D}) \subset Exc_2(\mathcal{C}, \mathcal{D}) \subset \cdots$$

Tayor/Goodwillie tower

Definition

The Taylor tower (or Goodwillie tower) of $F : \mathcal{C} \to \mathcal{D}$ at $X \in \mathcal{C}$ is the sequence of natural transformations of functors in $\mathcal{C}_{/X} \to \mathcal{D}$:

$$F \to \cdots \to P^X_{n+1} F \to P^X_n F \to \cdots \to P^X_1 F \to P^X_0 F \simeq F(X)$$

Can we recover the value $F(Y)$ from this sequence of approximations $P^X_n F(Y)$?

Definition (Converge)

The Taylor tower of $F : \mathcal{C} \to \mathcal{D}$ **converges** at $(Y \to X) \in \mathcal{C}_{/X}$ if the induced map

$$F(Y) \to holim_n P^X_n F(Y \to X)$$

is an equivalence in $\mathcal{D}$.

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Can we recover the value $F(Y)$ from this sequence of approximations $P_n^X F(Y)$?

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Convergence

Very general approaches to proving convergence seem rare, but Goodwillie has developed a set of tools based on connectivity estimates in the category of topological spaces and spectra.

Definition (stably n-excisive)

$F : \mathcal{C} \to \mathcal{D}$ is stably n-excisive, or satisfies stable n-th order excision, if the following is true for some numbers c and k:
If $X : N(P(S)) \to \mathcal{C}$ is any strongly coCartesian $(n+1)$-cube such that for all $s \in S$ the map $X(\emptyset) \to X(s)$ is k_s-connected and $k_s > k$, then the diagram $F \circ X$ is $(-c + \sum_{s \in S} k_s)$-Cartesian.
This condition is denoted as $E_n(c, k)$.

Definition (ρ-analytic)

$F : \mathcal{C} \to \mathcal{D}$ is ρ-analytic if there is some number q such that F satisfies condition $E_n(n\rho - q, \rho + 1)$ for all $n \geq 1$, i.e. it is stably n-excisive for all n where these connectivity estimates depend linearly on n with slope ρ.

Convergence

Theorem (Goodwillie)

Let $F : \mathcal{C} \to \mathcal{D}$ be a ρ-analytic functor where $\mathcal{C}$ and $\mathcal{D}$ are each either spaces or spectra. Then the Taylor tower of F at $X \in \mathcal{C}$ converges on those objects Y in $\mathcal{C}_{/X}$ whose underlying map $Y \to X$ is ρ-connected.

Example

The identity functor I on topological space Top is 1-analytic. This depends on higher dimensional versions of the Blakers-Massey theorem. Waldhausen's algebraic K-theory of spaces functor $A : Top \to Sp$ is also 1-analytic. The Taylor towers at $*$ of both of these functors converge on simply-connected spaces.

The Classification of Homogeneous Functors

Definition (layer)

The n-th **layer** of the Taylor tower of F at X is the functor $D_n^X F : \mathcal{C}_{/X} \to \mathcal{D}$ given by the **homotopy fiber** :

$$D_n^X F(Y) := hofib(P_n^X F(Y) \to P_{n-1}^X F(Y))$$

These layers play the role of homogeneous polynomials in the theory of calculus.

Definition (n-homogeneous)

Let $F : \mathcal{C} \to \mathcal{D}$ be a homotopy functor that admits n-excisive approximations, and where $\mathcal{D}$ has a terminal object. If n is a positive integer, we say that a functor $F : \mathcal{C} \to \mathcal{D}$ is n-reduced if $P_{n-1}F$ is a final object of $Exc_{n-1}(\mathcal{C}, \mathcal{D})$ (that is, if $(P_{n-1}F)(X)$ is a final object of $\mathcal{D}$, for each $X \in \mathcal{C}$). We will say that F is n-**homogeneous** if it is n-excisive and n-reduced.

The Classification of Homogeneous Functors

Proposition

The n-th layer of the Taylor tower is n-homogeneous.

Definition

A zero object of $\mathcal{C}$ is an object which is both initial and final. We will say that $\mathcal{C}$ is **pointed** if it contains a zero object.

Definition

An ∞-category $\mathcal{C}$ is **stable** if it satisfies the following conditions:

1. There exists a zero object $0 \in \mathcal{C}$.
2. Every morphism in $\mathcal{C}$ admits a fiber and a cofiber.
3. A triangle in $\mathcal{C}$ is a fiber sequence if and only if it a cofiber sequence.

The Classification of Homogeneous Functors

A triangle in $\mathcal{C}$ is a diagram:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

A triangle is a fiber sequence if it is a pullback square, and a cofiber sequence if it is a pushout square.

Any suitable pointed ∞-category $\mathcal{C}$ admits a stabilization, that is a stable ∞-category $Sp(\mathcal{C})$ together with an adjunction:

$$\Sigma^\infty_{\mathcal{C}} : \mathcal{C} \leftrightarrows Sp(\mathcal{C}) : \Omega^\infty_{\mathcal{C}}$$

which generalizes the suspension spectrum and infinite-loop space adjunction functors.

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The Classification of Homogeneous Functors

Proposition

If $L : \mathcal{C}^d \to \mathcal{D}$ is 1–excisive in each of the d variables, then the functor sending X to $L(X, X, \cdots, X)$ in $Fun(\mathcal{C}, \mathcal{D})$ is d excisive.

Theorem (Goodwillie)

Let $F : \mathcal{C} \to \mathcal{D}$ be an n-homogeneous functor between pointed ∞-categories. Then there is a symmetric multilinear functor $L : Sp(\mathcal{C})^n \to Sp(\mathcal{D})$, and a natural equivalence

$$F(X) \simeq \Omega^{\infty}_{\mathcal{D}}(L(\Sigma^{\infty}_{\mathcal{C}} X, \Sigma^{\infty}_{\mathcal{C}} X, \cdots, \Sigma^{\infty}_{\mathcal{C}} X)_{h\mathfrak{S}_n})$$

where we are taking the homotopy orbit construction with respect to the action of the symmetric group $\mathfrak{S}_n$.

Example

For functors $Top_* \to Top_*$, a symmetric multilinear functor is uniquely determined (on finite CW-complexes at least) by a single spectrum with a symmetric group action. Applying this classification to the layers of the Taylor tower of $F : Top_* \to Top_*$, we get an equivalence:

$$D_n F(X) \simeq \Omega^\infty(\partial_n F \wedge (\Sigma^\infty X)^{\wedge n})_{h\mathfrak{S}_n}$$

where $\partial_n F$ is a spectrum with an action of the symmetric group $\mathfrak{S}_n$. We call it the n**-th derivative** of F (at $*$).

Example

The n-th derivative of the functor $\Sigma^\infty\Omega^\infty : Sp \to Sp$ is equivalent to $\mathbb{S}^0$ with trivial $\mathfrak{S}_n$-action.

$$D_n(\Sigma^\infty\Omega^\infty)(X) \simeq X^{\wedge n}_{h\mathfrak{S}_n}$$

Definition

Let $\mathcal{P}_n$ be the poset of partitions of the set $1, \cdots, n$, ordered by refinement. Let $|\mathcal{P}_n|$ be the geometric realization of $\mathcal{P}_n$. Note that $\mathcal{P}_n$ has both an initial and a final object. It follows in particular that $|\mathcal{P}_n|$ is contractible. Let $\partial|\mathcal{P}_n|$ be the subcomplex of $|\mathcal{P}_n|$ spanned by simplices that do not contain both the initial and final element as vertices. Let $T_n = |P_n|/\partial|P_n|$.

Proposition

T_n is an $(n-1)$-dimensional complex with an action of the symmetric group $\mathfrak{S}_n$. T_n is equivalent to $\bigvee_{(n-1)!} S^{n-1}$ non-equivariantly. There is a $\mathfrak{S}_{n-1}$-equivariant equivalence $T_n \simeq \mathfrak{S}_{n-1_+} \wedge S^{n-1}$.

Theorem

There is a $\mathfrak{S}_n$-equivariant equivalence of spectra $\partial_n I = \mathbb{D}(T_n)$, where $\mathbb{D}$ is the Spanier-Whitehead dual.

Approximation of identity and periodic homotopy

Theorem (Arone-Mahowald)

Let X be an odd-dimensional sphere, and let p be a prime. The homology with mod p coefficients of the spectrum $\partial_n I \wedge X^{\wedge n}_{\mathfrak{S}_n}$ is non-trivial only if n is a power of p.

Theorem (Arone-Mahowald)

Let X be an odd-dimensional sphere, and work p-locally for a prime p. For $k \geq 0$, the map $X \to P_{p^k} I(X)$ is a v_k-periodic equivalence.

Theorem (Arone-Mahowald)

The cohomology of $\partial_n I \wedge X^{\wedge n}_{\mathfrak{S}_n}$ is free over A_{k-1} where A_k is the subalgebra of the Steenrod algebra generated by $\{Sq^1, Sq^2, Sq^4, \cdots, Sq^{2^k}\}$ for $p = 2$ and by $\{\beta, P^1, P^p, \cdots, P^{p^{k-1}}\}$ for $p > 2$.

Approximation of identity and periodic homotopy

Theorem (Arone-Mahowald)

Fix a prime p and localize at p. Let X be an even-dimensional sphere. Then $D_n I(X) \simeq *$ if n is not a power of p or twice a power of p, otherwise, $D_n I(X)$ has only p-primary torsion. The map $X \to P_{2p^k} I(X)$ is a v_k-periodic equivalence (i.e. induced v_k-periodic homotopy groups $v_k^{-1}\pi_*(-)$ are isomorphic).

Proposition (Bousfield-Kuhn functor reformulation)

Suppose $X = S^q$ is a sphere and localize at p. The natural transformation:

$$\Phi_n(X) \to \Phi_n(P_k I(X))$$

is an equivalence for q odd and $k = p^n$, or q even and $k = 2p^n$.

Approximation of identity and periodic homotopy

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Appendix: review of periodic homotopy

A finite complex V is said to be of type k if $K(n)_*V$ is trivial for $n < k$ and non-trivial for $n = k$. By the Periodicity Theorem, for each $k \geq 0$ there exists a finite complex of type k. Furthermore, suppose $k \geq 1$, and V_k is a complex of type k. Then there exists a self-map $f : \Sigma^{d|v_k|+i}V_k \to \Sigma^i V_k$ for some $i \geq 0$, $d \geq 1$, whose effect on $K(n)_*$ is an isomorphism for $n = k$ and zero for $n > k$. A map with these properties is called a v_k-periodic map. Any two v_k-periodic self maps of V_k are equivalent after taking some suspensions and iterations. We define the v_k-periodic homotopy groups of X with coefficients in V_k to be
$v_k^{-1}\pi_*(X; V_k) := v_k^{-1}[\Sigma^* V_k, X]_{Sp} = colim(\pi_*(X^V) \to \pi_{*+d|v_k|}(X^V) \to \cdots)$, which depend on the choice of V_k but do not depend on the self-map f.
Let $g : X \to Y$ be a map of spaces. If there exists one complex V_k of type k for which g induces an isomorphism $v_k^{-1}\pi_*(X; V_k) \to v_k^{-1}\pi_*(Y; V_k)$ then f induces an isomorphism for every such V_k.
$T(n)$ is defined to be $T(n) := hocolim(\Sigma^\infty V_k \to \Omega^{d|v_k|}\Sigma^\infty V_k \to \Omega^{2d|v_k|}\Sigma^\infty V_k \to \cdots)$
Functor $L_{T(n)}$ doesn't depend on the choise of V_k and the self map.
$v_k^{-1}\pi_*$-isomorphism $\Leftrightarrow$ $T(n)_*$-isomorphism $\Rightarrow$ $K(n)_*$-isomorphism

Appendix

The Bousfield-Kuhn functor Φ_k is a functor from pointed spaces to spectra, the functor is constructed as an inverse homotopy limit $\Phi_k(X) = holim_i \, v_k^{-1} X^{V_k^i}$, where $\{V_k^i\}$ is a direct system of complexes of type k with certain properties.
The main property of Φ_k is that there is an equivalence $\Phi_k(\Omega^\infty E) \simeq L_{T(k)}(E)$.
There is a variant of the Bousfield–Kuhn functor $\Phi_{K(n)} := L_{K(n)} \circ \Phi_n$. There is an equivalence $\Phi_{K(n)}(\Omega^\infty E) \simeq L_{K(n)}(E)$.