Periodicity in classical and motivic homotopy theory

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Chromatic homotopy theory and motivic homotopy theory

The chromatic approach to computing the homotopy groups of a finite complex X is recursive. Firstly, find a non-nilpotent self map $f : \Sigma^d X \to X$ and compute f-local part $\pi_*(X)[f^{-1}]$, next solve the problem of computing the f-torsion elements in $\pi_*(X)$ by replacing X with X/f and going back to the first step. The periodicity theorem of Devinatz-Hopkins-Smith ensures that this algorithm can proceed indefinitely without divergence. Because there always exists the next higher-level self-map, and it is unique up to self-composition.

Motivic homotopy theory was developed by Morel and Voevodsky to import homotopy theory techniques into algebraic geometry. Since Voevodsky use this conception to resolve the Milnor conjecture and Bloch–Kato conjecture, lots of mathematicians continue to advance motivic homotopy theory, eventually enabling it to benefit algebraic topology and facilitate computations of classical homotopy groups.

Exotic periodicity

The τ-periodic $\mathbb{C}$-motivic stable homotopy groups are isomorphic to the classical stable homotopy groups with $\tau^{\pm}$ adjoined. i.e. $\pi_{*,*}S^{0,0}[\tau^{-1}] \cong \pi_* S^0[\tau^{\pm}]$. This comparison between the $\mathbb{C}$-motivic and classical situations is induced by the Betti realization functor that takes a complex variety to its underlying topological space of $\mathbb{C}$-valued points.

The Hopf map $\eta \in \pi_1(S^0)$ admits a lift $\eta \in \pi_{1,1}(S^{0,0})$ and this class is not nilpotent. Thus, the classical Nishida nilpotence theorem, which asserts that positive-degree elements of $\pi_*(S^0)$ are nilpotent, fails in a direct motivic analogue. There are at least two different non-nilpotent self-maps 2 and η. Since the first step in the chromatic approach already fails, the process has to be refined, and that the linear ordering of periodicities v_n has to be replaced by a more complex structure.

Exotic periodicity

This non-nilpotent behavior led Miller to propose that there might be an infinite family of periodicity operators w_n such that $w_0 = \eta$, in analogy to the v_n-periodicity operators that begin with the non-nilpotent element $v_0 = 2$. This guess turned out to be correct because Andrews construct a self map $w_1^4 : \Sigma^{20,12} S/\eta \to S/\eta$ by lifting element from Adams spectral sequence of $End(S/2)$ to motivic Adams-Novikov spectral sequence of $End(S/\eta)$, and Gheorghe construct motivic fields $K(w_n)$ designed to detect w_n-periodic phenomena by obstruction theory.

Krause found more exotic $\mathbb{C}$-motivic periodicities β_{ij} corresponding to the elements h_{ij} of the May spectral sequence. The v_n-periodicities correspond to the elements $\beta_{n+1,0}$, while the w_n-periodicities correspond to $\beta_{n+1,1}$. In this talk, we'll briefly introduce the result of Krause.

Vanishing lines and self-maps from comodule categories

Gheorghe-Wang-Xu has shown that there is an equivalence of stable ∞-categories between the bounded derived category of p-completed BP_*BP-comodules that are concentrated in even degrees, and the category of motivic left module spectra over $S^{0,0}/\tau$. i.e.$Mod_{S/\tau} \cong Comod^{cg,even}_{D(BP_*BP)}$.

BP_*BP admits a quotient Hopf algebra P_*, which can be identified with a certain subalgebra of the dual Steenrod algebra A_*. Krause obtained vanishing lines and self-maps informations by describing P_* through a sequence of extensions by particularly small Hopf algebras. Inductively, one can see that only specific slopes of vanishing lines are possible,i.e. $d_{ij} = \frac{1}{p^{j+1}(p^i-1)-1}$, and that self-maps parallel to the minimal vanishing line always exist.

By the GWX-Theorem, the statements about vanishing lines and self-maps in the category $Comod^{cg}_{D(BP_*BP)}$ immediately carry over to the category $Mod_{S/\tau}$, and then lifting them from S/τ-modules to all motivic spectra via a τ-Bockstein spectral sequence. In general, $S^{n,w}/\tau \in Mod_{S/\tau}$ corresponds to $S^{n,2w-n} \in Comod^{cg}_{D(BP_*BP)}$.

Periodicity β_{ij}

For each $i > j \geq 0$, there is a $\mathbb{C}$-motivic p-complete cellular S/τ-module $K(\beta_{ij})$, with homotopy groups isomorphic to $\pi_{**}(K(\beta_{ij})) \simeq \mathbb{F}_p[\alpha_{ij}, \beta_{ij}^{\pm}]/\alpha_{ij}^2$, with $|\alpha_{ij}| = (2p^j(p^i - 1) - 1, p^j(p^i - 1))$, and $|\beta_{ij}| = (2p^{j+1}(p^i - 1) - 2, p^{j+1}(p^i - 1))$. The slope of β_{ij} in the motivic (n, w)-grading is $d_{ij}^{mot} = \dfrac{p^{j+1}(p^i - 1)}{2p^{j+1}(p^i - 1) - 2}$.

For X a compact, p-complete cellular motivic spectrum over $\mathbb{C}$, the homotopy groups $\pi_{n,w}(X)$ admit a minimal vanishing line in (n, w)-grading. The slope of such a minimal vanishing line coincides with one of the d_{ij}^{mot} for some $i > j \geq 0$, characterized as the largest d_{ij}^{mot} for which $K(\beta_{ij})_{**}X \neq 0$. There is a non-nilpotent self-map $\Sigma^{|\theta|}X \xrightarrow{\theta} X$ of slope d_{ij}^{mot}, which induces an isomorphism on $K(\beta_{ij})_{**}X$.

Further problems I

The $\beta_{i,j}$-chromatic theory analogous to the classical chromatic homotopy theory is still very unclear and not fully established. There are the following significant differences between motivic and classical chromatic phenomenon.

The exotic K-theories $K(\beta_{ij})$ are not field spectra, so many of the usual nice properties of Morava K-theories don't carry over to them. It is not clear whether the thick subcategories characterized by vanishing of a single $K(\beta_{ij})$ are actually prime thick subcategories. This is equivalent to the question of whether $K(\beta_{ij})_{**}(X \bigotimes Y) = 0$ implies $K(\beta_{ij})X = 0$ or $K(\beta_{ij})Y = 0$ for compact X and Y . And we also don't know whether the thick subcategories characterized by vanishing of $K(\beta_{ij})$ including $K(n)$ and $K(w_n)$ form a complete list of prime thick subcategories of finite *p*-complete cellular motivic spectra.

Motivic spectra seem to typically have additional self-maps of lower slope. The easiest example is on the sphere $S^{0,0}$. At $p = 2$, the minimal vanishing line of the sphere has slope 1, with corresponding $\beta_{1,0}$ self-map η, but the sphere also admits an $\beta_{2,1}$ self-map $\bar{\kappa}_2 \in \pi_{44,24}(S^{0,0})$.

There is a v_1^4-self-map $\theta : \Sigma^{8,4} S/2 \to S/2$, on $S/2$, any product $\eta^a \theta^b$ for $a > 0,\ b > 0$ is not nilpotent. $\eta^a \theta^b$ has slope $\frac{a+4b}{a+8b}$ and acts trivially on all the K-theories $K(\beta_{ij})_{**}(S/2)$ and $K(n)_{**}(S/2)$. They provide examples of non-nilpotent self-maps of slope d_{ij}^{mot} for any $i > j \geq 0$ that are not β_{ij} self-maps.

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Thank You!